Problem 12a

Let a be a partially ordered set and X and Y be **non-empty** subsets of A with least upper bounds and greatest lower bounds. If $X \subseteq Y \subseteq A$ then $glb(Y) \leq glb(X) \leq lub(X) \leq lub(Y)$

Proof:

If a is any lower bound for Y then $a \leq y \; \forall \; y \in Y$. But since all elements of X are also elements of Y this means that $a \leq x \; \forall \; x \in X$ and a is a lower bound for X. Since glb(Y) is a lower bound for Y, it is also a lower bound for X. Since glb(X) is the greatest lower bound for X it is greater than or equal to any lower bound for X so, in particular: $glb(Y) \leq glb(X)$. A similar argument implies that $lub(X) \leq lub(Y)$. Finally, since X is not empty $\exists \; x \in X$, and by the definition of lower bound, $glb(X) \leq x$. By the definition of upper bound $x \leq lub(X)$. Putting the two inequalities together gives $glb(X) \leq x \leq lub(X)$, so $glb(X) \leq lub(X)$. Thus, since all three inequalities hold:

$glb(Y) \leq glb(X) \leq lub(X) \leq lub(Y)$